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The generation of combustion noise by chemical inhomogeneities in steady, low-Mach-number duct flows

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The low-frequency character of two model problems is exploited in order to illustrate the acoustic consequences of the interactions between chemically reacting (or relaxing) inhomogeneities and flames or constrictions in ducts. The monopole of the former is associated with heat transfer in a fluid which exhibits variations in its specific heats, while in the latter there is an extension of the classical phenomenon associated with the pulsations of an inhomogeneity of the fluid compressibility. This second mechanism is found to be insignificant, but the heat-conduction source is considered to be very powerful at sufficiently low Mach numbers; in fact, to first order it is independent of the flow Mach number for laminar, as well as a certain class of turbulent, flows.

1. Introduction

Since Lighthill's (1952) original contribution to the theory of aerodynamic sound, in which attention was paid primarily to the role of fluctuations in turbulent stresses, considerable effort has been expended over experimental and theoretical assessments of the role of various other sources of sound, collectively described as 'excess noise' (see, for example, Crighton 1975 for a review of the aeroacoustics of inert flows). Underlying these efforts is the fact that, given an acoustic field scaling on the mean-flow Mach number (denoted by M) as M^ν, $\nu > 0$, then for $M < 1$ a reduction in ν corresponds to an increase in emitted power. Excess noise sources which are regarded as significant include those associated with density inhomogeneities (e.g. Morfey 1973; Howe 1975*b*), vortical inhomogeneities (Howe & Liu 1977) and thermo-viscous dissipation (e.g. Crighton 1975; Kempton 1976). Combustion noise, which is a major consequence of exothermic reacting flows, has been the subject of a number of papers during the 1960's and 1970's, although some of the fundamental concepts were foreshadowed in a paper by Chu (1955).

This paper aims to highlight the acoustic ramifications of two monopole mechanisms, namely the interaction of reacting (or relaxing) chemical inhomogeneities with (i) mean temperature gradients and (ii) mean pressure gradients.

(i) *Temperature gradients.* A considerable number of papers have been published on combustion noise (see, for example, Chiu & Summerfield 1974; Strahle 1975), and most, if not all, of these papers have been concerned with concentration fluctuations in free turbulence and/or in enclosures. Whilst it is widely appreciated that non-uniform mixing of reactants is liable to lead to additional generation of sound, no attempt has been made to date to quantify the process, and the present paper aims to fill this gap, in § 3.

Howe (1975*a, b*) invoked reciprocity in order to deal with the interaction of density inhomogeneities with mean pressure gradients, and the present analysis adapts that procedure in order to cope with the interaction of *chemical* inhomogeneities with mean *temperature* gradients. The model problem envisaged herein involves a blob convecting at low Mach number towards a plane, steady flame in a rigid, constant-area duct (which possesses adiabatic walls). Mass diffusion and viscosity are neglected, whereas heat conduction and chemical reactions are accounted for; this is justifiable in view of the fundamental processes being addressed in this paper, namely chemical reactions and heat transfer. Upstream the blob differs from the ambient fluid only in its concentrations and its density, although, like the pressure and temperature, the latter may be specified to be uniform too if a diluent is present in the system. As the blob sweeps through the flame, the temperature at its centre differs from that of the background flow, and heat is therefore transferred across its surface; since the specific heats change discontinuously across the blob's surface, this process represents a monopole (Mamayev 1975; Kempton 1976). The blob is assumed to be small on the scale of the flame thickness and, after obtaining a scaling law for the sound propagated downstream in the laminar case, conjectures are made as to the probable scaling for turbulent flows and the influence of a duct end; it transpires that this mechanism can be powerful, particularly for laminar flows and for a certain class of turbulent flows.

(ii) *Pressure gradients*. It is well known (Rayleigh 1945; Strahle 1976) that an inhomogeneity in γ (the ratio of specific heats) generates monopole sound when it is subjected to pressure fluctuations. Ffowcs Williams & Howe (1975) and Whitfield & Howe (1976) have also studied this mechanism, and in §4 we extend that work to account for chemical reactions or relaxation in the inhomogeneity.

The model problem which has been chosen involves a steady, chemically frozen, low-Mach-number irrotational background flow through a rigid duct which possesses constant cross-sectional area except for an isolated constriction. Approaching the neck is a small, sharp-fronted chemical inhomogeneity which contains a reacting (or relaxing) gas (molecular transport is ignored). Upstream, the density of the blob may be specified to be identical to that of the ambient fluid, so that no dipole would be generated during the passage through the neck. As the inhomogeneity sweeps through the neck it pulsates in response to the ambient pressure, and the analysis facilitates an assessment of the role of the non-equilibrium phenomenon. It appears that within this linearized treatment (the reaction time is assumed constant) the reaction term is unlikely to be more powerful than the classical frozen one.

2. The acoustic analogy

Howe (1975*b*) proposed an acoustic analogy which is well suited for various types of problems, including those involving source–mean-flow interactions. That analogy is useful in the present context as well, although it needs to be generalized to cope with thermal and/or chemical changes.

The differential entropy relation (e.g. de Groot & Mazur 1969; Clarke & McChesney 1976) encompassing irreversible processes is

$$T\,dS = dh - dP/\rho - \sum_{i=1}^{n} A_i\,d\sigma_i, \tag{2.1}$$

where A_i is a 'chemical potential' associated with the non-equilibrium variable σ_i (when considering chemical reactions, σ_i could represent a mass or mole fraction, with

$n+1$ species existing in the system), T is the translational temperature, S and h are the specific entropy and enthalpy, P is the pressure and ρ the density. We ignore viscous dissipation (whilst admitting thermal conduction), and Crocco's equation then reads

$$\frac{\partial \mathbf{v}}{\partial t}+\nabla h_0 = T\nabla S-\boldsymbol{\omega}\times\mathbf{v}+\sum_{i=1}^{n} A_i\nabla\sigma_i, \tag{2.2}$$

where h_0 is the stagnation enthalpy and $\mathbf{v}$ is the mass-averaged velocity. Invoking (2.2), as well as the continuity equation in the form

$$\frac{1}{\rho a_f^2}\frac{DP}{Dt}+\nabla.\mathbf{v} = -\hat{Q} \tag{2.3}$$

here
$$\rho\hat{Q} = \left(\frac{\partial\rho}{\partial S}\right)_{P,\sigma}\frac{DS}{Dt}+\sum_{i=1}^{n}\left(\frac{\partial\rho}{\partial\sigma_i}\right)_{P,S,\sigma_j}\frac{D\sigma_i}{Dt} = \frac{D\rho}{Dt}-\frac{1}{a_f^2}\frac{DP}{Dt} \tag{2.4}$$

and
$$a_f^2 = (\partial P/\partial\rho)_{S,\sigma} \tag{2.5}$$

(a_f is the frozen sound speed), leads to the following generalization of Howe's equation:

$$\left[\frac{D}{Dt}\left(\frac{1}{a_f^2}\frac{D}{Dt}\right)+\frac{1}{a_f^2}\frac{D\mathbf{v}}{Dt}.\nabla-\nabla^2\right]h_0 = \left(\frac{1}{a_f^2}\frac{D\mathbf{v}}{Dt}-\nabla\right).\left(T\nabla S-\boldsymbol{\omega}\times\mathbf{v}+\sum_1^n A_i\nabla\sigma_i\right)$$
$$+\frac{D}{Dt}(\hat{R}/a_f^2)-\frac{\partial\hat{Q}}{\partial t}+\mathbf{v}.\left[\nabla\left(\frac{1}{\rho a_f^2}\frac{\partial P}{\partial t}\right)-\frac{\partial}{\partial t}\left(\frac{1}{\rho a_f^2}\nabla P\right)\right], \tag{2.6}$$

where
$$\hat{R} = T\frac{DS}{Dt}+\sum_1^n A_i\frac{D\sigma_i}{Dt}, \tag{2.7}$$

and we note that in an acoustic field $P' \sim \rho_0 h_0'$ for negligible mean flow Mach number. If the mixture is thermally perfect, the term in square braces on the right-hand side of (2.6) becomes

$$\mathbf{v}.\left[\frac{\partial}{\partial t}(\ln P)\,\nabla(\gamma_f^{-1})-\nabla(\ln P)\frac{\partial}{\partial t}(\gamma_f^{-1})\right],$$

where γ_f is the frozen ratio of specific heats. At a first glance it might appear that this term could represent the classical γ monopole because of the appearance of derivatives of γ; however, this is not so, and indeed the term is nonlinear. The quantity $\hat{Q}$ is immediately recognizable as a monopole; in fact, the relevance of $\rho^{-1}D\rho/Dt$ was pointed out some time ago by Chu (1955), and Morfey (1976) and Kempton (1976) used this term in connexion with diffusive effects. It is interesting, too, that Ffowcs Williams' (1974) expression for free space radiation contains the term

$$4\pi|\mathbf{x}|\,P' \simeq -\rho_0\frac{\partial}{\partial t}\int\left[\frac{1}{\rho(1-M_r)^2}\frac{D\rho}{Dt}\right]^{\#} d^3y,$$

where $\#$ denotes retarded time, M_r is the source Mach number relative to the observer; M_r may be neglected for low-Mach-number flows, resulting in direct correspondence between the two approaches.

3. 'Temperature forcing'

In this section we address the problem of a chemical inhomogeneity approaching a steady, plane premixed flame (a deflagration) situated in a constant-area duct, the latter having adiabatic walls. By invoking the energy equation, it can be shown that

$$\hat{Q} = \psi\left(\frac{\partial q_j}{\partial x_j} - \Phi\right) + \frac{1}{\rho}\sum_{i=n}^{n} Q_i \frac{D\sigma_i}{Dt}. \tag{3.1}$$

where

$$\left.\begin{aligned} \psi &= \frac{\alpha_f}{\rho C_{P_f}}, \quad \alpha_f = \frac{-1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{P,\sigma}, \\ Q_i &= \left(\frac{\partial \rho}{\partial \sigma_i}\right)_{P,T,\sigma_j} + \rho^2\psi\left(\frac{\partial h}{\partial \sigma_i}\right)_{P,T,\sigma_j}. \end{aligned}\right\} \tag{3.2}$$

Here Φ is the viscous dissipation, q_j is the heat flux $-k\,\partial T/\partial x_j$, C_P is the constant pressure specific heat, and the subscripts f and e denote frozen and equilibrium conditions respectively.

Observing that flame speeds are normally very small (e.g. Lewis & von Elbe 1938, pp. 160 and 206; Williams 1965, p. 99), we may neglect terms in (2.6) which involve the flow Mach number, and indeed we may regard the background state as one of constant pressure. Linearizing (2.6) about that state yields

$$\left[\nabla^2 - \frac{1}{(a_f^2)_0}\frac{\partial^2}{\partial t^2}\right]\frac{P'}{\rho_0} = \frac{\partial \hat{Q}}{\partial t}. \tag{3.3}$$

The subscript zero denotes background values. Obviously, other source terms are present in (2.6), but $\hat{Q}$ is the only one of current interest to us. As a word of caution, we should perhaps mention that, whilst Howe (1975*b*) has retained ∇S in favour of $\boldsymbol{\omega}$ and vice versa, Crocco's theorem is frequently interpreted as an indication of the production of vorticity when there are gradients in entropy and/or concentration (Liepmann & Roshko 1957), so that it may not be possible to consider these sources independently. We will ignore Φ and concentrate on the role of heat conduction and chemical reactions.

The essence of this paper's simplification of $\hat{Q}$ lies in the neglect of mass diffusion. This permits us to assume that an inhomogeneity approaching the flame possesses concentrations different from those in the surrounding gas, and that the concentrations change discontinuously at the blob's surface. This technique was used by Howe (1975*b*), who assumed discontinuous density and temperature in order to analyse the behaviour of 'entropy spots'. In the present context the blob's density could be different from that of the ambient fluid, but in any case the radiated dipole (which arises from the acceleration of the blob relative to its environment during its passages through mean pressure gradients) would be negligible because the background pressure is virtually constant. All we insist on is that far upstream the pressure and temperature are uniform, and the blob need only be identifiable by its density and/or composition. We could in fact arrange for the density to be uniform too by altering the concentrations of a diluent, as may be seen by considering the equation of state of a perfect gas mixture:

$$P = \rho RT; \quad R = \mathscr{R}\sum_{i=1}^{n+1}\sigma_i/W_i, \quad \sum_{i=1}^{n+1}\sigma_i = 1, \tag{3.4}$$

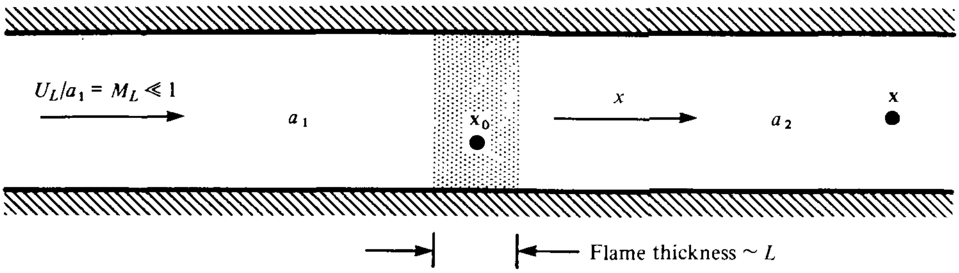

FIGURE 1. Steady flame in a constant-area duct. The duct walls are rigid and insulated.

($\mathscr{R}$ is the universal gas constant and W_i is a molecular weight); this would exclude the above-mentioned dipole totally. Be that as it may, whereas the dipole considered by Morfey (1973), Ffowcs Williams & Howe (1975) and Howe (1975*b*) relies on 'forcing' by mean pressure gradients, the present mechanism involves interaction with mean temperature gradients.

Now write

$$\left.\begin{aligned}\sigma_i &= \sigma_{i1}(\mathbf{x},t)\,H(-f)+\sigma_{i2}(\mathbf{x},t)\,H(f),\\ \psi &= \psi_1(\mathbf{x},t)\,H(-f)+\psi_2(\mathbf{x},t)\,H(f),\end{aligned}\right\} \tag{3.5}$$

where $f(\mathbf{x},t) = 0$ is a material surface defining the inhomogeneity, $f < 0$ in the blob's interior and H is the Heaviside function. Hence

$$\frac{D\sigma_i}{Dt} = H(-f)\frac{D\sigma_{i1}}{Dt}+H(f)\frac{D\sigma_{i2}}{Dt}.$$

In this paper we wish to concentrate on the dissipative heat-conduction term in (3.1), and we may therefore ignore $D\sigma_i/Dt$, or alternatively we could insist on the concentration gradients in the blob faithfully following those of the background flame. Equation (3.3) therefore becomes

$$\left[\nabla^2-\frac{1}{(a_f^2)_0}\frac{\partial^2}{\partial t^2}\right]\frac{P'}{\rho_0} \simeq \frac{\partial}{\partial t}\left(\psi\frac{\partial q_j}{\partial x_j}\right). \tag{3.6}$$

This equation will now be solved asymptotically by invoking reciprocity in order to construct a low-frequency Green's function. Landau & Lifshitz (1959, p. 288) prove reciprocity for the wave equation in free space, in the presence of spatial variations in the sound speed. It is easily shown that reciprocity still holds in the presence of solid bodies when a simple impedance condition is satisfied at their surfaces (in particular, rigid surfaces are permitted).

Consider the time-harmonic problem represented by

$$[\nabla^2+k_0^2(x)]\frac{P'}{\rho_0} = \delta(x-x_0), \tag{3.7}$$

$$k_0(x) = \omega/a_{f_0}(x). \tag{3.8}$$

Then, by the reciprocal theorem, the disturbance produced at the observation point $\mathbf{x}$ (see figure 1) by a point source at $\mathbf{x}_0$ is equal to the disturbance produced at $\mathbf{x}_0$ by the same source at $\mathbf{x}$. At this stage we restrict the size of the blob to be small on the scale of the flame thickness L, and the radiated field will possess wavelengths large compared with L (viz. at low Mach numbers the source is compact); the low-frequency character of the problem is embodied in the small parameter $\epsilon = k_2 L$, where $k_2 = \omega/a_2$. Laminar flame thicknesses are typically of the order of a millimetre or a fraction thereof, but in any case the general conclusions of this paper do not hinge on the fact that the blob is small compared to the flame thickness, and similar results may be deduced for large 'slugs' of chemically inhomogeneous material. We ignore all modes other than the plane (Morse & Ingard 1968, p. 500), so the reciprocal wave incident on the flame is

$$\frac{i}{2k_2 s}\exp(ik_2|x_0-x|),$$

where s is the cross-sectional area of the duct. Furthermore, it is shown in the appendix that in the vicinity of the flame the reciprocal disturbance $G = P'/\rho_0(x_0)$, to $O(1)$, is

$$G(\mathbf{x}_0, \mathbf{x}; \omega) = \frac{ia_2}{\omega s(1+\alpha)}[1+O(\epsilon)]\exp(ik_2|x_0-x|),$$

where $\alpha = a_2/a_1$. The Green's function is thus obtained by reversing $\mathbf{x}_0$ and $\mathbf{x}$, multiplying by $(2\pi)^{-1}\exp[i\omega(\tau-t)]$ and integrating with respect to ω alongt he real axis. The contour must be indented above the pole at the origin in order to satisfy the causality requirements, and we find that

$$G(\mathbf{x}, \mathbf{x}_0; t, \tau) = \frac{a_2}{s(1+\alpha)} H\left(t-\tau-\frac{|x-x_0|}{a_2}\right). \tag{3.9}$$

A similar expression may be found for observation points upstream of the flame.

The sound radiated downstream is obtained by convolving the right-hand side of (3.6) with G:

$$P' \simeq \frac{\rho_2 a_2}{s(1+\alpha)}\iint \frac{\partial}{\partial \tau}\left(\psi\frac{\partial q_j}{\partial y_j}\right) H\left(t-\tau-\frac{|x-y_1|}{a_2}\right) d^3\mathbf{y}\, d\tau. \tag{3.10}$$

Performing the τ integration first, we get

$$\begin{aligned} P' &\simeq \frac{\rho_2 \alpha_2}{s(1+\alpha)}\iint \delta\left(t-\tau-\frac{|x-y_1|}{a_2}\right)(\psi_1-\psi_2)\, q_j\, \delta(f)\frac{\partial f}{\partial y_j}\, d^3\mathbf{y}\, d\tau \\ &= \frac{-\rho_2 a_2}{s(1+\alpha)}\left\{[\psi]\int_\Sigma q_j n_j\, d^2\mathbf{y}\right\}^{\#}, \end{aligned} \tag{3.11}$$

where Σ denotes the surface of the inhomogeneity, n_j is its outward unit normal vector, and $\#$ denotes evaluation at the retarded time. This is immediately recognized as the monopole associated with the transfer of heat between fluids possessing different specific heats. We emphasize that specific heats of gas mixtures can vary not only with temperature, but with composition too.

Let us now derive scale laws for (3.11). If we assume that the mixture is perfect, and denote the jump in the value of a quantity across the surface $f = 0$ by square braces, then

$$[\psi] = \frac{-1}{P}\left[\frac{1}{\gamma_f}\right].$$

The thickness of the thermal layer surrounding the inhomogeneity is characterized by $(\lambda L/U_L)^{\frac{1}{2}}$, where $\lambda\ (= k/\rho C_{Pf})$ is the thermal diffusivity and U_L is the laminar flame speed; we then find that, for laminar flows,

$$P'_{\text{lam}} \sim \frac{\rho_2 a_2^2}{\alpha(1+\alpha)}\left(\frac{l^2}{s}\right)\left(\frac{\gamma_*}{\gamma_*-1}\right)\left[\frac{1}{\gamma_f}\right]_*\left(\frac{\Delta T}{T}\right)_* P_{eL*}^{-\frac{1}{2}} M_L, \tag{3.12}$$

where l is a characteristic linear dimension of the blob, $P_{eL} = LU_L/\lambda$, ΔT is the temperature difference between the centre of the blob and its environment, and $*$ denotes a typical mean value. It is noteworthy that P' scales on $U_L^{\frac{1}{2}}$, but what is of greater significance is the fact that this field depends on the flame speed U_L, which is an intrinsic property of the fluid dependent *only* on the chemical state upstream; in other words, this field is independent of U, where U is the mean flow speed. If U differs from U_L, the flame is no longer stationary in our reference frame, but the speed of the inhomogeneity relative to the flame will always be U_L, and it is this fact which is responsible for the U^0 dependence. Even if U vanishes identically, a flame propagating through a stationary combustible mixture will generate sound whenever it encounters an inhomogeneity. Of course, if the flame is moving in our frame of reference, we may expect Doppler factors to arise, but terms of order M have been neglected relative to one anyway.

Naturally, our model is an idealized one; in reality laminar flames are seldom planar, boundary layers exist at the duct walls and, indeed, flame propagation in ducts is a subject in its own right (see Lewis & von Elbe 1938; Guénoche 1964; Borisov 1978). Nevertheless it is unlikely that the arguments presented above will fail if the inhomogeneity is small relative to the duct diameter and if it is not located close to the walls.

Turning now to turbulent flows, we must differentiate between two regimes (e.g. Libby & Williams 1976): (i) wrinkled laminar flames; (ii) distributed reaction models. In the former model the combustion region is envisaged as a thin flame which is distorted by the flow and which propagates locally at the laminar flame speed normal to itself; evidently, the scaling (3.12) still applies, since we expect the blob's velocity component along the normal to the flame front to be roughly equal to U_L.

$$P'_{\mathrm{WLF}} \simeq P'_{\mathrm{lam}}, \tag{3.13}$$

where WLF stands for 'wrinkled laminar flame'. For model (ii), we can no longer associate a thin flame with the reaction zone, and indeed we can only say that the frequencies scale on U/D (where $s \propto D^2$) and that the thermal layer thickness scales on $(\lambda D/U)^{\frac{1}{2}}$. Furthermore, if we employ Kempton's (1976) arguments, which imply that the surface area of the blob scales not on l^2 but on $P_e^{\frac{1}{2}} l^2$ (assuming that the Prandtl number is $O(1)$), we find that

$$P'_{\mathrm{DRM}} \sim \frac{\rho_2 a_2^2}{\alpha(1+\alpha)}\left(\frac{l^2}{s}\right)\left(\frac{\gamma_*}{\gamma_*-1}\right)\left[\frac{1}{\gamma_f}\right]_*\left(\frac{\Delta T}{T}\right)_* M, \tag{3.14}$$

where $M = U/a_2$: Hence this field scales on U, and as M decreases to zero we expect a transition towards M^0 (of (3.13)) to arise.

4. 'Pressure forcing'

This section deals with the influence of chemical reactions on the pulsations of an inhomogeneity which encounters mean pressure gradients. For the purpose of illustration, consider first the scattering of an acoustic field in a reacting (or relaxing) medium by a small chemical inhomogeneity (mass and heat diffusion are ignored). It is well known that a compact monopole radiates into free space according to

$$P'_s \simeq \frac{\rho_0}{4\pi r}\left(\frac{d\zeta}{dt}\right)^{\#}, \quad \zeta = dV/dt, \tag{4.1}$$

where V is the volume of the source. If we restrict our analysis to small disturbances and consider the density ρ to be a function of P, S and σ, then in the absence of molecular transport the perturbations in entropy are of second order (Clarke & McChesney 1976, p. 184), and hence

$$\frac{1}{V_0}\frac{dV}{dt} = \frac{-1}{\rho_0 a_{f_0}^2}\frac{dp}{dt} - \left(\frac{Q}{\rho}\right)_0 \frac{d\sigma}{dt}, \tag{4.2}$$

where

$$\left.\begin{aligned} Q &= \left(\frac{\partial\rho}{\partial\sigma}\right)_{P,T} + \frac{\rho\alpha_f}{C_{Pf}}\left(\frac{\partial h}{\partial\sigma}\right)_{P,T} \\ &\equiv \left(\frac{\partial\rho}{\partial\sigma}\right)_{P,S} + \frac{\rho\alpha_f}{C_{Pf}}\left(\frac{\partial h}{\partial\sigma}\right)_{P,S}, \end{aligned}\right\} \tag{4.3}$$

and the subscript zero denotes the background state (assumed to be one of equilibrium). Invoking the rate equation

$$\hat{\tau}_0 \frac{d\sigma}{dt} + \sigma = \sigma_e(P, S) \tag{4.4}$$

and the identity

$$(Qa_f^2)_e \left(\frac{\partial\sigma_e}{\partial P}\right)_S = \hat{\delta} = (a_f^2)_e/a_e^2 - 1, \tag{4.5}$$

it follows that

$$\sigma' = \left(\frac{\hat{\delta}}{\hat{\tau}Qa_f^2}\right)_0 \int_0^\infty P'(t-\tau)\exp\left(-\tau/\hat{\tau}\right)d\tau \tag{4.6}$$

and that

$$\frac{1}{V_0}\frac{dV}{dt} = -\left(\frac{1}{\rho a_f^2}\right)_0 \frac{dP}{dt} - \left(\frac{\hat{\delta}}{\hat{\tau}\rho a_f^2}\right)_0 \int_0^\infty \frac{dP}{dt}(t-\tau)\exp\left(-\tau/\hat{\tau}\right)d\tau. \tag{4.7}$$

Evidently, if the background state is spatially uniform, (4.7) describes the local (linear) effects of compressibility in a reacting medium, and scattering of the acoustic field will occur only if the wave encounters inhomogeneities in $1/(\rho a_f^2)_0$ and $(\hat{\delta}/\hat{\tau}\rho a_f^2)_0$, in which case the effective source strength is given by

$$\frac{1}{V_0}\frac{dV}{dt} = \left[\frac{1}{\rho a_f^2}\right]_0 \frac{dP}{dt} + \int_0^\infty \frac{dP}{dt}(t-\tau)\left[\frac{\hat{\delta}\exp\left(-\tau/\hat{\tau}\right)}{\hat{\tau}\rho a_f^2}\right]_0 d\tau, \tag{4.8}$$

where for any variable ψ, $[\psi] = \psi_2 - \psi_1$, the subscripts 1 and 2 denoting the interior and exterior of the blob respectively. For example, consider the case $\hat{\tau}$ = constant throughout a perfect gas:

$$P_0\zeta/V_0 = \left[\frac{1}{\gamma_f}\right]_0 \frac{DP}{Dt} - \left[\frac{1}{\gamma_e} - \frac{1}{\gamma_f}\right]_0 F(t) \tag{4.9}$$

where

$$F(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\omega P'(\omega)\, e^{-i\omega t}\, d\omega}{i\omega\hat{\tau} - 1}, \tag{4.10a}$$

$$P'(\omega) = \int_{-\infty}^{\infty} P'(t)\, e^{i\omega t}\, dt. \tag{4.10b}$$

Clearly, the first term is the familiar classical monopole (Rayleigh 1945; Whitfield & Howe 1976) associated with changes in γ_f, and the second term is a chemical contribution associated with changes in the difference between γ_f and γ_e. If

$$P'(t) = P\, e^{i\omega_0 t}, \tag{4.11}$$

then

$$F(t) = \frac{-i\omega_0 P'}{i\omega_0\hat{\tau} + 1}$$

and
$$-\frac{1}{V_0}\frac{d\zeta}{dt} = \omega_0^2 P'(t)\left\{\left[\frac{1}{\rho a_f^2}\right]_0 + \left[\frac{\hat{\delta}}{\rho a_f^2}\right]_0 (i\omega_0 \hat{\tau}+1)^{-1}\right\}. \quad (4.12)$$

Equation (4.12) acquires the expected forms in the limits $\omega_0\hat{\tau} \to 0$ and $\omega_0\hat{\tau} \to \infty$. Generally, we may state that, when $\hat{\tau}$ = constant and heat conduction is ignored, chemical effects are no stronger than γ effects when the 'forcing' is due to pressure.

Now let us turn to the problem of a small reacting inhomogeneity convecting through a low-Mach-number, irrotational frozen duct flow. Hence we write

$$\left(\frac{Q}{\rho}\right)_0 = H(-f)\left(\frac{Q}{\rho}\right)_0, \quad (4.13)$$

where $f(\mathbf{x}, t) = 0$ describes the inhomogeneity's material surface, and $f < 0$ in the blob's interior. The acoustic process is therefore described by

$$\left[\frac{1}{a_0^2}\left(\frac{\partial}{\partial \tau} + U_j \frac{\partial}{\partial y_j}\right)^2 - \nabla^2\right] h_0 \simeq -\left[\frac{1}{\rho a_f^2}\right]_0 \frac{\partial}{\partial \tau}\left[H(-f)\frac{DP}{D\tau}\right]$$
$$-\left(\frac{\hat{\delta}}{\hat{\tau}\rho a_f^2}\right)_0 \frac{\partial}{\partial \tau}\left\{H(-f)\int_0^\infty \frac{DP}{D\tau}(y, \tau-\alpha)\exp(-\alpha/\hat{\tau})\, d\alpha\right\}. \quad (4.14)$$

The radiation produced by low-Mach-number convection through a neck in a duct of otherwise uniform cross-section can be determined by means of the renormalized, low-frequency Green's function (Howe 1975*a*):

$$G(X, Y; t, \tau) \simeq \frac{a_0}{2s} H\left\{t-\tau-\frac{|X_1-Y_1|}{a_0[1+M\,\mathrm{sgn}\,(X_1-Y_1)]}\right\}, \quad (4.15)$$

where $Y_1 = \Phi(\mathbf{y})$ is a potential describing irrotational flow through the neck, such that $\Phi \sim y_1$ as $|y_1| \to \infty$, $X_1 = \Phi(\mathbf{x})$, and s is the asymptotic duct cross-sectional area. Write

$$\hat{T}(X_1 Y_1, t) = t - \frac{|X_1-Y_1|}{a_0[1+M\,\mathrm{sgn}\,(X_1-Y_1)]}, \quad T_1 = \hat{T}-\tau, \quad (4.16)$$

and denote the position of the centre of the inhomogeneity by $\mathbf{x}_0(t)$; then

$$\frac{P'}{\rho_0} \simeq -\iint \frac{\partial}{\partial \tau} H(-f)\left\{\left[\frac{1}{\rho a_f^2}\right]_0 \frac{DP}{D\tau} + \left[\frac{\sigma}{\hat{\tau}\rho a_f^2}\right]_0\right.$$
$$\left.\times \int_0^\infty \frac{DP}{D\tau}\,(y, \tau-\alpha)\exp(-\alpha/\hat{\tau})\, d\alpha\right\}\frac{a_0}{2s} H(\hat{T}-\tau)\, d^3\mathbf{y}\, d\tau$$
$$= \frac{V_0 a_0}{2s}\int_{-\infty}^\infty \left\{\left[\frac{1}{\rho a_f^2}\right]_0 \frac{DP}{D\tau}[\mathbf{x}_0(\tau)] + \left[\frac{\hat{\delta}}{\hat{\tau}\rho a_f^2}\right]_0\right.$$
$$\left.\times \int_0^\infty \frac{DP}{D\tau}[\mathbf{x}_0(\tau-\alpha)]\exp(-\alpha/\hat{\tau})\, d\alpha\right\}\delta(T_1^*-\tau)\frac{\partial T_1^*}{\partial \tau}\, d\tau, \quad (4.17)$$

where V is the inhomogeneity's volume, $T_1^* = T^*-\tau$, and

$$T^* = \hat{T}\{X_1[\mathbf{x}_0(\tau)], Y_1[\mathbf{x}_0(\tau)], t\}. \quad (4.18)$$

Hence

$$\frac{P'}{\rho_0} \simeq \frac{V_0 a_0}{2s}\left\{\left[\frac{1}{\rho a_f^2}\right]_0 \frac{DP}{D\tau}[\mathbf{x}_0(T)] + \left[\frac{\hat{\delta}}{\hat{\tau}\rho a_f^2}\right]_0 \int_0^\infty \frac{DP}{D\tau}[\mathbf{x}_0(T-\alpha)]\exp(-\alpha/\hat{\tau})\, d\alpha\right\}, \quad (4.19)$$

where $\tau = T$ is the zero of T_1^*, i.e.

$$T \simeq t - |x_1|/a_0(1+M\,\mathrm{sgn}\,x_1). \quad (4.20)$$

Note that, since the background flow is assumed to be chemically frozen, $\hat{\tau}$ jumps across the blob's surface from its non-infinite value in the interior to its infinite value exterior to the blob.

It is easy to show that (4.19) regains the appropriate forms in the equilibrium and frozen limits. For the sake of completeness, the scaling of the ultimate term in (4.19) is given below.

(i) Equilibrium limit ($U\hat{\tau}/L \to 0$): the relevant values of α lie in the range 0 to $\hat{\tau}$, and

$$P' \sim \rho_0 a_0^2 \left(\frac{V_0}{sL}\right)\left[\frac{\hat{\delta}}{\gamma_f}\right]_0 M^3. \tag{4.21}$$

(ii) Frozen limit ($U\hat{\tau}/L \to \infty$): α lies in the range 0 to L/U, leading to

$$P' \sim \rho_0 a_0^2 \left(\frac{V_0}{sL}\right)\left[\frac{\hat{\delta}}{\gamma_f}\right]_0 \left(\frac{L}{U\hat{\tau}}\right) M^3. \tag{4.22}$$

Here the field scales on U^2, but is in fact weaker than the M^3 field associated with the first term in (4.19) because of the appearance of the small number $L/U\hat{\tau}$. Comparison of these fields with that radiated by the dipole associated with density inhomogeneities (Howe 1975*b*) shows that the present mechanism is weaker than the dipole by a factor no larger than $O(M)$.

5. Conclusions

Two model problems have been studied in order to illustrate the acoustic effects associated with the interactions of chemical inhomogeneities with mean temperature gradients ('temperature forcing') and mean pressure gradients ('pressure forcing').

In the former case it is found that the heat conduction term is responsible for a powerful field in the duct scaling on M^0, for laminar flows and even for some turbulent flows (viz. those flows in which a turbulence scale is much larger than the flame thickness, modelled as 'wrinkled laminar flames'). It is argued that if the turbulence scale is not considerably larger than the flame thickness ('distributed reaction model'), the field downstream will scale on M. It should be pointed out that dimensional arguments indicate that inclusion of the effects of mass diffusion results in the multiplication of the scaling laws by a Lewis–Semenov number.

In order to place the importance of this duct pressure field in perspective, let us make an order-of-magnitude comparison with two other sources of 'excess noise', namely those associated with density and vortical inhomogeneities. The former (e.g. Howe 1975*b*) scales on $[\rho]\, l^2 D^{-2} U^2$, and the latter (Howe & Liu 1977) on $\rho_0 D^{-1}[D]\, Uv$, where D is a characteristic cross-sectional dimension of the duct, $[D]$ is a measure of the change in cross-sectional size at the neck, and v is a measure of the perturbation velocity associated with the vorticity wave approaching the constriction. We consequently conclude that P'_{lam} will dominate these two fields when

$$M \lesssim \left\{\left[\frac{1}{\gamma_f}\right]\frac{\rho_2 [T]\, M_L}{[\rho]\, T_2 P_{eL}^{\frac{1}{2}}}\right\}^{\frac{1}{2}} \tag{5.1a}$$

and

$$M \lesssim \left(\frac{l}{D}\right)^2 \left[\frac{1}{\gamma_f}\right] \frac{D[T]\, M_L}{[D]\, m T_2 P_{eL}^{\frac{1}{2}}}, \quad m = v/a_2. \tag{5.1b}$$

Similarly, P'_{DRM} will overwhelm these fields when

$$M \lesssim \left[\frac{1}{\gamma_f}\right]\frac{\rho_2[T]}{[\rho]T_2} \tag{5.2a}$$

and

$$M \lesssim \left(\frac{l}{D}\right)^2\left[\frac{1}{\gamma_f}\right]\frac{[T]D}{T_2[D]}. \tag{5.2b}$$

We reiterate that it is possible for $[\rho]$ to vanish when $[T]$ is non-zero, and the density-inhomogeneity source disappears under such circumstances. Obviously, there is a delicate balance reflected in these expressions, but on the basis of the conclusions reached by Howe & Liu (1977) we may state that, generally, the vortical source is more important than the density source. Naturally, the Mach numbers indicated by (5.1) and (5.2) are likely to be small under typical turbojet conditions, and the present noise mechanism will assert itself only at low power settings.

The pressure forcing field generally appears to be no more powerful than $O(M^3)$ in the duct and this mechanism is therefore weaker than the two other excess noise sources.

As regards the transmission of these fields from a duct end, it should be remembered that the present analysis is based on the assumption that k_2D is small. If mean flow is ignored, the transmitted field includes an additional factor of M over and above the scaling laws given herein (Rayleigh 1945, p. 196; Noble 1956, p. 115); the influence of mean flow is negligible at low Mach numbers (Munt 1977, Howe 1979, Cargill 1979).

As far as 'direct' combustion noise is concerned, the agreement between experiment and theory is rather unsatisfactory at present. It seems that the initial procedure we could undertake now is a comparison of the power of the present source mechanism transmitted from a duct end with theoretical and experimental relations for the free-space 'direct' acoustic radiation emitted by an open combustion zone.

Equations (3.13), (3.14) show that the present transmitted power scales on $P_{eL}^{-\frac{1}{2}}M_L^{\frac{3}{2}}M^2$ and M^4 for the wrinkled flame and distributed reaction models respectively. Bragg's (1963) expression (for wrinkled flames) scales on M_LM^3, and when $M \lesssim M_L$ the present source is strong compared with the *free-space* direct field (Strahle 1975 quotes experimental correlations containing the terms $M_L^{1\cdot 83}M^{2\cdot 67}$).

An alternative assessment would compare the direct free-space power with that emitted by the present mechanism in *free space*. The latter scales on $P_{eL}^{-1}M_L^4$ and M^4 for the two models respectively; it is significant that the wrinkled model again results in a field which is independent of U. In any case, it appears that in the context of both comparisons the present source can dominate direct combustion noise, but only at very low Mach numbers, of the order M_L, and it will begin to dominate the excess noise associated with density and vortical inhomogeneities at Mach numbers which are greater than this small value.

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Appendix. Low-frequency scattering by the flame

Consider a time-harmonic propagating wave approaching a premixed flame (of thickness L small compared with the wavelength) from the right (see figure 1). In view of (2.6), the acoustic field is regarded as a process governed by the equation

$$\phi_{xx}+k_0^2(x/L)\,\phi = 0, \quad \phi = P'/\rho_0(x/L), \tag{A 1}$$

and the incident field is e^{-ik_2x} (the time-dependence will be omitted henceforth). We now assume that k_0 asymptotes its values k_1 and k_2 *exponentially*, and we transform (A 1) as follows:

$$x' = k_2 x, \quad \lambda = k_0/k_2, \quad \epsilon = k_2 L, \quad \alpha = k_1/k_2 > 1. \tag{A 2}$$

Omitting the primes,

$$\phi_{xx} + \lambda^2(x/\epsilon)\,\phi = 0. \tag{A 3}$$

Evidently, the process is characterized by an outer region in which k_0 is virtually constant and an inner region where variations in k_0 are significant. The volume of literature on sound speed variations in ducts is considerable (see Nayfeh 1973, p. 308) but attention has mainly been paid to the *high*-frequency case. The present analysis is similar to the model problems considered by Lesser & Crighton (1975), and it utilizes the method of matched asymptotic expansions (e.g. Van Dyke 1975).

Outer region

Assuming an outer expansion in the form

$$\phi = \begin{cases} e^{-ix} + \sum_{j=0}^{\infty} g_j(\epsilon)\,\phi^{(j)}(x), & x > 0, \\ \sum_{j=0}^{\infty} g_j(\epsilon)\,\phi^{(j)}(x), & x < 0, \end{cases} \tag{A 4}$$

we find that $g_0 = 1$, $g_1 = \epsilon$, $g_2 = \epsilon^2$ and, in view of the exponential behaviour of λ,

$$\phi^{(j)}(x) = \begin{Bmatrix} R_j\, e^{ix} \\ T_j\, e^{-i\alpha x} \end{Bmatrix} \quad \text{all } j; \tag{A 5}$$

R_j and T_j will be determined after the interaction with the flame has been analysed.

Inner region

Transform (A 1):

$$\phi_{XX} + \epsilon^2\lambda^2(X)\,\phi = 0, \quad X = x/\epsilon. \tag{A 6}$$

Writing

$$\phi = \sum_{j=0}^{\infty} G_j(\epsilon)\,\Phi_j(X), \tag{A 7}$$

we find that $G_0 = 1$, $G_1 = \epsilon$, $G_2 = \epsilon^2$, and that

$$\Phi_0 = AX + B, \quad \Phi_1 = CX + D, \qquad \text{(A 8), (A 9)}$$

$$\frac{d^2\Phi_2}{dX^2} = -\lambda^2(X)\,\Phi_0. \tag{A 10}$$

Matching the outer series to $O(1)$ to the inner series to $O(1)$ gives

$$A = 0, \quad 1 + R_0 = B = T_0. \tag{A 11}$$

Matching the outer series to $O(1)$ to the inner series to $O(\epsilon)$ gives

$$-i(1 - R_0) = C = -i\alpha T_0, \tag{A 12}$$

i.e.

$$T_0 = \frac{2}{1+\alpha}, \quad R_0 = \frac{1-\alpha}{1+\alpha}. \tag{A 13}$$

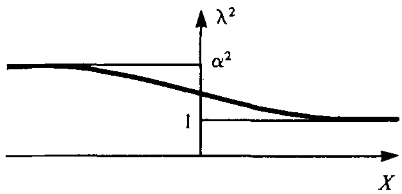

FIGURE 2

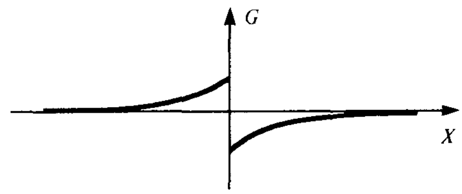

FIGURE 3

FIGURE 2. Typical profile of $\lambda(X)$.
FIGURE 3. Typical profile of $G(X)$.

Matching the outer series to $O(\epsilon)$ to the inner series to $O(\epsilon)$ gives

$$R_1 = D = T_1. \tag{A 14}$$

In order to calculate Φ_2 without choosing a specific profile for λ, let us define λ as follows:

$$\lambda^2 = \alpha^2 H(-X) + H(X) - G(X), \tag{A 15}$$

where H is the Heaviside function. This extracts the behaviour of λ at $\pm\infty$, thereby permitting us to match in some generality; typical profiles are drawn in figures 2 and 3. Matching the outer series to $O(\epsilon)$ to the inner to $O(\epsilon^2)$ then yields

$$R_1 = T_1 = D = \frac{-2i\psi_\infty}{(1+\alpha)^2}, \tag{A 16}$$

where

$$\psi_\infty = \int_{-\infty}^{\infty} G(y)\, dy. \tag{A 17}$$

At a first glance it seems surprising that the outer field is described by reflexion and transmission coefficients which differ from the classical expressions (Rayleigh 1945, p. 78), where density ratios appear in addition to sound speed ratios. This apparent contradiction is compounded when one attempts to integrate the Helmholtz equation across a fixed discontinuity, because one discovers that the results are altered by a change in the dependent variable. If we denote values just to the left and right of the discontinuity by the subscripts $-$ and $+$ respectively, then we find that employing the *pressure* as the dependent variable leads to the condition

$$\left(\frac{\partial P}{\partial x}\right)_+ = \left(\frac{\partial P}{\partial x}\right)_- \tag{A 18}$$

or

$$\rho_- U_- = \rho_+ U_+, \tag{A 19}$$

which has the appearance of a mass-continuity condition. Invoking this relation immediately leads to (A 13).

If the *velocity potential* is used as the dependent variable, (A 18) is replaced by

$$\left(\frac{\partial \phi}{\partial x}\right)_+ = \left(\frac{\partial \phi}{\partial x}\right)_-, \tag{A 20}$$

which of course leads to the classical Rayleigh results.

The correct jump conditions are, in fact, deduced after integration of the continuity equation. Denoting the position of the discontinuity by $X(t)$, the required relation is not unfamiliar (Sirovich 1968):

$$\rho_+ U_+ - \rho_- U_- = (\rho_+ - \rho_-)\frac{dX}{dt}. \tag{A 21}$$

It is clear that the Rayleigh results are recovered when the discontinuity is a *material* discontinuity, for then $U_+ = U_- = dX/dt$ and (A 21) is satisfied identically. If, however, material flows *through* the discontinuity, as is the case when the discontinuity is a flame, (A 20) is inappropriate, and (A 19) applies if we neglect flame movement (as we indeed have done in this paper). It is interesting that (A 13) agrees with the low-Mach-number limit of the results of Dowling (1979), who analysed a different problem utilizing a wholly dissimilar technique.

Note too that, if λ is monotonic, we can always find a position for the origin (the 'centre' of the flame) which yield $\psi_\infty = 0$, i.e. (A 13) applies, at least to $O(\epsilon)$. If, however, the temperature profile exhibits an overshoot, then ψ_∞ will not vanish for *any* choice of the origin.

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